**Chapter 5 Summary**

m = mass of spring (units: kg are mass, pounds need to be divided by 32)

k = spring constant (lb/feet)

s = amount stretched

Hooke’s Law: the spring exerts a force proportional to the amount it is stretched: mg=ks

**5.1.1 Free undamped motion summary:**

mx” + kx = 0

x” + ω2x = 0 where ω2 = k/m

x = c1 cosωt + c2 sinωt

ω/2 π =frequency

**5.1.2 Free Damped Motion**

Damping is proportional to a power of the velocity.

Assume power = 1.

mx” = -kx – βx’

mx” + βx’ + kx = 0

Alternate form: x” + 2λx’ + ω2x = 0 (λ is used for algebraic convenience)

Case 1 λ2 - ω2 > 0 (overdamped) // will cross equilibrium at most once

Case 2 λ2 - ω2 = 0 (critically damped) // will cross equilibrium at most once

Case 3 λ2 - ω2 < 0 (underdamped) // will oscilate

**5.1.3 Spring/Mass Systems: Driven Motion**

Suppose an external force, f(t), acts on the spring

mx” + βx’ + kx = f(t)

x” + 2λx’ + ω2x = F(t)

**2 Linear Models: Boundary Value Problems**

 Deflection of a beam

--------------------------------------------------

Assume a beam of length L with uniform cross sections

y(x) = distance it sags

 EI y(4) = w(x)

(E is Young’s modulus of elasticity, I is moment of inertia of a cross section, w(x) is weight per unit length at x)

Assume a uniform weight w(x) = w0 per unit length

EIx(4) = w0

Solution: y(x) = c1 + c2 x + c3 x2+ c3 x3  + w0/24EI x4

Boundary conditions are determined by how the ends are supported.

Embedded at both ends (ends are rigid): y(0)= y(L) =y’(0)= y’(L) = 0

Hanging free at L: y(0) = y’(0) = 0, y”(L)= y”’(L) = 0

Simply supported: y(0)= y(L) =y”(0)= y”(L) = 0

**Chapter 5 Modeling with higher order differential equations (MORE DETAILED)**

m = mass of spring (units: kg are mass, pounds need to be divided by 32)

k = spring constant (lb/feet)

s = amount stretched

Hooke’s Law: the spring exerts a force proportional to the amount it is stretched: mg=ks

**5.1.1 Free Undamped Motion**

x(t) = displacement at time t, x’ = velocity, x” = acceleration

Equation of motion: mx” + kx = 0

Alternate form: x” + ω2x = 0 where ω2 = k/m

Solution: x(t) =c1 cosωt + c2 sinωt

Period: T = 2π/ω

Frequency: f = ω/2 π (# cycles per second)

Example: A mass weighing 4 pounds stretches a spring 6 inches. At t = 0 the spring is released from 9 inches with no imparted velocity. Find the equation of motion.

m = W/g = 4/32 //convert from weight to mass

s = 6 = .5 ft //convert to feet

k = mg/s = (4/32)32 /.5 = 2

mx” + kx=0

4/32 x” +2x=0

x” + 16x = 0

x = c1 cos4t + c2 sin4t

x(0) = ¾ => ¾ = c1

x’(0) = 0

x’ = -c1 4sin 4t + c2 4cos4t

0 = 4c2

x = ¾ cos4t

Suppose it is released with an initial upward velocity of 1 ft/second

x = c1 cos4t + c2 sin4t

x(0) = ¾ , x’(0) = -1

As before c1= ¾

-1 = c1

x =- ¾ cos4t - sin 4t

Example: When a 10 k mass is attached to a spring, the frequency of simple harmonic motion is observed to be 3/ π cycles per second. Find the equation of motion.

Frequency is ω/2 π = 3/ π

ω = 6

x” + 36 x = 0

x = c1 cos 6t + c2 sin6t

What is the spring constant?

ω2 = k/m

k = ω2 m = 36x10 = 360

What is the frequency if the mass is replaced by a 30k mass?

ω2 m = 360

ω2 = 360/m = 360/30 = 12

Frequency is ω/2 π = 12/2 π = 6/ π

Example: A 20 pound weight stretches a spring 6 inches. It is released from 12 inches below equilibrium. Find the equation of its motion.

x(0) = 1, x’(0) = 0

m = 20/32 = 5/8

k needs to be lb/foot 20/.5 = 40

5/8x” + 40 x = 0

x” + 64 x = 0

x = c1 cos 8t + c2 sin8t

Using x(0) = 1 = c1

Using x’(0) = 0 = -8c1 sin 0 + 8c2 cos 0, c2 = 0

x = cos 8t

How would this change if it were given an initial upward velocity of 3 ft/sec2?

x’(0) = -3

**5.1.2 Spring/Mass Systems: Free Damped Motion**

Undamped motion is unrealistic

Damping is proportional to a power of the velocity.

Assume power = 1.

mx” = -kx – βx’

Equation of motion: mx” + βx’ + kx = 0

Alternate form: x” + 2λx’ + ω2x = 0 (λ is used for algebraic convenience)

2λ = β/m

ω2 = k/m

m = -λ +/- sqrt(λ2 - ω2)

x = e- λt (c1 e^sqrt(λ2 - ω2)t + c2 e^-sqrt(λ2 - ω2)t)

Case 1 λ2 - ω2 > 0 (overdamped) // will cross equilibrium at most once

Case 2 λ2 - ω2 = 0 (critically damped) // will cross equilibrium at most once

Case 3 λ2 - ω2 < 0 (underdamped) // will oscilate

Example A 32 pound mass stretches a spring 32/9 feet, Assume damping is equal to 6 times the instantaneous velocity. The weight is pulled down 10 feet and released. Find the equation of motion.

m = 32/32

k = 32/ (32/9) = 9

x” + 6x’ + 9x = 0

x(0) = 10, x’(0) = 0

(λ2 - ω2  = 9 – 9 =0, critically damped)

Example

A 32 pound mass stretches a spring 4 feet, Assume damping is equal to 2 times the instantaneous velocity. The weight is pulled down 10 feet and released.

m = 32/32 = 1, k = 32/4 = 4

x” + 2x’ + 4x = 0

 (λ2 - ω2  = 1 – 4 <0 underdamped)

**5.1.3 Spring/Mass Systems: Driven Motion**

Suppose an external force, f(t), acts on the spring

mx” + βx’ + kx = f(t)

x” + 2λx’ + ω2x = F(t)

Example:

A force cos t is applied to the object in the last example

x” + 2x’ + 4x = cos t

xp = Asint + B cost

-Asint –Bcost + 2(Acost –Bsint) + 4(Asint + B cost) = cost

...

**5.2 Linear models: Boundary Value Problems**

Another example where linear DEs come up

Deflection of a beam

--------------------------------------------------

Assume a beam of length L with uniform cross sections

 Axis of symmetry

In real life it bends: y(x) = distance it sags

Resulting equation after a bit of analysis:

EI y(4) = w(x) where E is Young’s modulus of elasticity of the material, I is moment of inertia of a cross section and w(x) is weight per unit length at x

Boundary conditions are determined by how the ends are supported.

Assume a uniform weight w(x) = w0 per unit length

EIx(4) = w0

Solution to the homogeneous equation: y(x) = c1 + c2 x + c3 x2+ c3 x3

Find a particular solution yp = A x4:

AEi(4)(3)(2) = w0

A = w0/24EI

y(x) = c1 + c2 x + c3 x2+ c4 x3 + w0/24EI x4

Example: a beam embedded at both ends (ends are rigid).

y(0)= y(L) =y’(0)= y’(L) = 0

(instantaneous change in x = 0 because of rigidity)

How would this change if the beam hangs free at L?

y(0) = y’(0) = 0, y”(L)= y”’(L) = 0

(you don’t know what y(L) or y’(L) is but since it is hanging freely, assume nothing contributes to a change in y” or y’’’ at L)

How would this change if the beam were simply supported?

Remove y’() = 0, add y”()= 0

y(0)= y(L) =y”(0)= y”(L) = 0